Stability of a conducting rotating fluid of variable density

By S. P. TALWAR
Physics Department, Delhi University, Delhi, India

(Received 29 June 1960)

The Rayleigh instability of an incompressible, infinitely conducting, inviscid fluid of variable density is investigated under the influence of an horizontal magnetic field and coriolis forces. After establishing the equations of the problem when both the density and the magnetic field vary with distance in the upward direction, two special cases of density distribution are studied in detail. Both the magnetic field and coriolis forces are found to have a stabilizing influence on the configuration and further it is concluded that they may bring about stability in the configuration when it is thoroughly unstable without them.

1. Introduction

The problem of the hydromagnetic stability of conducting fluid of variable density is of considerable importance in astrophysics (e.g. in theories of sunspot magnetic fields, heating of solar corona, stability of the stellar atmospheres in magnetic field) and has been investigated in recent years by some workers (Hide 1955; Ferraro & Plumpton 1958). Talwar (1959) investigated the influence of a magnetic field on the character of the equilibrium of two superposed highly conducting fluids and of a stratified layer of highly conducting fluid. It was shown that the magnetic field has a stabilizing influence on the configuration and helps in bringing about stability in a configuration when it is thoroughly unstable without a field. However, in many astrophysical and geophysical problems, coriolis forces also play an important role and their effect may at least be as important as those of electromagnetic forces. Keeping this in mind, we investigate in the present paper the effect of a magnetic field on the equilibrium of an inviscid, incompressible, infinitely conducting *rotating* fluid of variable density. The density stratification may be supposed to be due to a change in composition. In §2 we shall establish the general equations of the problem, assuming that there is a variation of both density and the horizontal magnetic field with respect to distance in the upward direction alone. For simplicity, the fluid is assumed to be incompressible, inviscid, and of infinite electrical conductivity, so that the magnetic field assumed is effectively 'frozen' in the fluid. However, in a more realistic situation one should study the problem including the effects of viscosity, compressibility, and finite conductivity, and it is hoped to include them in a later communication.

Earlier, Lord Rayleigh (1883) initiated the study of the hydrodynamic stability of a fluid of variable density. Having developed the equations for a horizontally stratified, non-rotating, inviscid, incompressible fluid, he discussed two special density variations corresponding to (*a*) two superposed fluids and (*b*) a continuously stratified fluid. Bjerknes, Bjerknes, Solberg & Bergeron (1933) extended Rayleigh's treatment for superposed fluids to include the effect of rotation. Further the investigation was extended by Chandrasekhar (1955) and Hide (1956) to include viscosity in the non-magnetic case without and with rotation, respectively.

2. Equations of the problem

Take axis $Oxyz$ such that Oz is vertical. Suppose that there exists a horizontal magnetic field (stratified upwards) along the x-direction in the fluid which is further assumed to have a variable density in the upward direction. Let the configuration rotate uniformly with an angular velocity Ω about the z-axis.

The equation of motion appropriate to the problem under consideration is,

$$\rho\frac{\partial \mathbf{u}}{\partial t}+\rho(\mathbf{u}.\nabla)\,\mathbf{u}-2\rho(\mathbf{u}\times\mathbf{\Omega}) = -\nabla p+\mathbf{g}\rho+\frac{\mu}{4\pi}[(\nabla\times\mathbf{H})\times\mathbf{H}], \tag{1}$$

where $\mathbf{u}, \mathbf{H}$ denote the velocity and the magnetic field vectors and p, ρ, g respectively denote the pressure, density at a point and the acceleration due to gravity with component $-g$ in the z-direction (or g may denote the net acceleration downward in case there is an additional imposed acceleration over and above the usual gravitational acceleration). μ is the permeability of the medium.

The equation of continuity of matter is

$$\frac{\partial \rho}{\partial t}+\nabla.(\rho\mathbf{u}) = 0. \tag{2}$$

Since the fluid is heterogenous and incompressible and the diffusion effects are ignored, the density of an element does not alter as the element moves about; (2) then reduces to

$$\nabla.\mathbf{u} = 0. \tag{3}$$

For a perfectly conducting fluid which is also incompressible we have

$$\frac{\partial \mathbf{H}}{\partial t}+(\mathbf{u}.\nabla)\,\mathbf{H} = (\mathbf{H}.\nabla)\,\mathbf{u}. \tag{4}$$

Finally we have the equation

$$\nabla.\mathbf{H} = 0. \tag{5}$$

In order to investigate the stability of the static equilibrium configuration characterized by $\mathbf{u} = 0$, let us consider the effect of a small velocity field disturbance $\mathbf{u}$, with components u, v, w in x-, y- and z-directions, respectively. We write,

$$\left.\begin{aligned}\rho &= \rho_0(z)+\delta\rho(x,y,z,t),\\ p &= p_0(z)+\delta p(x,y,z,t),\\ \mathbf{H} &= \mathbf{H}_0(z)+\mathbf{h}(x,y,z,t),\end{aligned}\right\} \tag{6}$$

where $\delta\rho$, δp, and $\mathbf{h}$ are perturbations of the first order of smallness so that powers higher than the first and their mutual products can be ignored. Hence the equations for the first-order disturbance components, with an undisturbed horizontal field in the x-direction, can be written as

$$\rho_0 \frac{\partial u}{\partial t} - 2\rho_0 v\Omega = -\frac{\partial \delta p}{\partial x} + \frac{\mu}{4\pi} h_z \frac{dH_0}{dz}, \tag{7}$$

$$\rho_0 \frac{\partial v}{\partial t} + 2\rho_0 u\Omega = -\frac{\partial \delta p}{\partial y} + \frac{\mu}{4\pi} H_0 \left(\frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y}\right), \tag{8}$$

$$\rho_0 \frac{\partial w}{\partial t} = -\frac{\partial \delta p}{\partial z} - g\delta\rho - \frac{\mu}{4\pi} H_0 \left(\frac{\partial h_x}{\partial z} - \frac{\partial h_z}{\partial x}\right) - \frac{\mu}{4\pi} h_x \frac{dH_0}{dz}, \tag{9}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{10}$$

$$\frac{\partial \delta\rho}{\partial t} + w\frac{d\rho_0}{dz} = 0, \tag{11}$$

$$\frac{\partial h_x}{\partial t} = H_0 \frac{\partial u}{\partial x} - w\frac{dH_0}{dz}, \tag{12}$$

$$\frac{\partial h_y}{\partial t} = H_0 \frac{\partial v}{\partial x}, \tag{13}$$

$$\frac{\partial h_z}{\partial t} = H_0 \frac{\partial w}{\partial x}, \tag{14}$$

and

$$\frac{\partial h_x}{\partial x} + \frac{\partial h_y}{\partial y} + \frac{\partial h_z}{\partial z} = 0. \tag{15}$$

Suppose, as usual in problems of this kind, that the components of the disturbance vary with x, y, z, t as

$$(\text{some function of } z).\exp(ik_x x + ik_y y + nt),$$

where k_x and k_y are the horizontal wave-numbers of the harmonic disturbance. On substituting in the above equations (7)–(15) we get

$$n\rho_0 u - 2\rho_0 v\Omega = -ik_x \delta p + (\mu/4\pi) h_z DH_0, \tag{16}$$

$$n\rho_0 v + 2\rho_0 u\Omega = -ik_y \delta p + (\mu H_0/4\pi)(ik_x h_y - ik_y h_x), \tag{17}$$

$$n\rho_0 w = -D\delta p - g\delta\rho - (\mu/4\pi) H_0 [Dh_x - ik_x h_z + h_x(DH_0/H_0)], \tag{18}$$

$$ik_x u + ik_y v + Dw = 0, \tag{19}$$

$$n\delta\rho + wD\rho_0 = 0, \tag{20}$$

$$nh_x = H_0 ik_x u - wDH_0, \tag{21}$$

$$nh_y = H_0 ik_x v, \tag{22}$$

$$nh_z = H_0 ik_x w, \tag{23}$$

and

$$ik_x h_x + ik_y h_y + Dh_z = 0, \tag{24}$$

where D stands for d/dz.

We now eliminate some of the variables and derive an equation for, say, the component w of velocity perturbation vector. Multiply (16), (17) by ik_x and ik_y, respectively, and add, to get, on making use of (19),

$$-n\rho_0 Dw - 2\rho_0 \Omega(ik_x v - ik_y u) - \frac{\mu}{4\pi} ik_x h_z DH_0 \tag{25}$$

$$-\frac{\mu}{4\pi} H_0 ik_y [ik_x h_y - ik_y h_x] = k^2 \delta p, \tag{26}$$

where $k^2 = k_x^2 + k_y^2$. Eliminate δp from equations (25) and (18) (after having eliminated $\delta\rho$ by using equation (20)) by operating the former by D and multiplying the latter by k^2 and adding. In this way we get

$$-nD(\rho_0 Dw) - 2\Omega D[\rho_0(ik_x v - ik_y u)] - \frac{\mu}{4\pi} ik_x D(h_z DH_0) - \frac{\mu}{4\pi} ik_y D[H_0(ik_x h_y - ik_y h_x)] + n\rho_0 k^2 w + \frac{\mu H_0}{4\pi} k^2 \times \left[Dh_x - ik_x h_z + h_x \frac{DH_0}{H_0}\right] - \frac{gk^2}{n} wD\rho_0 = 0. \quad (27)$$

If for a moment, we assume $H = 0$ and write the equation for w, we obtain,

$$\left[\rho_0 nk^2 - \frac{gk^2}{n} D\rho_0\right] w - nD(\rho_0 Dw) = 2\Omega D(\rho_0 \zeta). \quad (28)$$

where we have written,

$$\zeta = ik_x v - ik_y u. \quad (29)$$

In equation (27), the terms involving the magnetic field reduce to (after some simplications employing equation (24))

$$\frac{\mu}{4\pi} ik_x[H_0(D^2 - k^2) h_z - h_z D^2 H_0]. \quad (30)$$

On making use of (23) we get the magnetic terms, after some simplifications, as,

$$-\frac{\mu}{4\pi n} k_x^2[H_0^2(D^2 - k^2) w + 2H_0 DwDH_0]. \quad (31)$$

Thus the complete equation (27) for w can be written as,

$$\left[\rho_0 nk^2 - \frac{gk^2}{n} D\rho_0\right] w - nD(\rho_0 Dw) - \frac{\mu k_x^2}{4\pi n}[H_0^2(D^2 - k^2) w + 2H_0 DH_0 Dw] = 2\Omega D(\rho_0 \zeta). \quad (32)$$

Now eliminate δp from equations (16) and (17) by multiplying the former by $-ik_y$ and latter by ik_x and adding. We get

$$n\rho_0[ik_x v - ik_y u] + 2\rho_0 \Omega[ik_y v + ik_x u] - (\mu/4\pi)[ik_x H_0(ik_x h_y - ik_y h_x) - ik_y DH_0 . h_z] = 0. \quad (33)$$

Thus the equation determining ζ becomes,

$$\zeta = \frac{2\rho_0 \Omega Dw}{\left[n\rho_0 + \frac{\mu H_0^2}{4\pi n} k_x^2\right]}. \quad (34)$$

On eliminating ζ in equations (32) and (34), we get the equation for w as

$$(D^2 - k^2) u \left[1 + \frac{\mu H_0^2}{4\pi r_c} \frac{k_x^2}{n^2}\right]^3 + \frac{4\Omega^2}{n^2} D^2 w \left[1 + \frac{\mu H_0^2}{4\pi\rho_0} \frac{k_x^2}{n^2}\right] + \frac{D\rho_0}{\rho_0}\left[\left(Dw + \frac{gk^2}{n^2} w\right)\left(1 + \frac{\mu H_0^2}{4\pi\rho_0} \frac{k_x^2}{n^2}\right)^2 + \frac{4\Omega^2}{n^2} Dw\left(1 + \frac{\mu H_0^2}{2\pi\rho_0} \frac{k_x^2}{n^2}\right)\right] + \frac{\mu k_x^2}{2\pi n^2 \rho_0} H_0 DH_0 Dw\left[\left(1 + \frac{\mu H_0^2}{4\pi\rho_0} \frac{k_x^2}{n^2}\right)^2 - \frac{4\Omega^2}{n^2}\right] = 0. \quad (35)$$

In §§ 3 and 4 we shall be studying two special cases, namely, the hydromagnetic stability of (i) a continuously stratified layer of fluid, and (ii) a configuration of two superposed fluids, respectively.

3. Hydromagnetic stability of a continuously stratified fluid of finite depth

Here we shall consider the special case in which an inviscid, infinitely conducting layer of fluid is confined between two rigid horizontal boundaries at $z = 0$, and $z = l$, and there is present a horizontal magnetic field in the x-direction. We assume that the density is stratified according to the law

$$\rho_0(z) = \rho_1 e^{\beta z}, \quad 0 \leqslant z \leqslant l, \\ = 0 \quad \text{elsewhere}, \tag{36}$$

where ρ_1, and β are constants.

We further assume, for reasons of simplification in analysis, that the permanent horizontal magnetic field is also stratified in the upward direction in such a way that the local hydromagnetic wave velocity is constant throughout the fluid, which means that H_0^2/ρ_0 is constant throughout the fluid or

$$H_0^2(z) = H_1^2 e^{\beta z}. \tag{37}$$

With equations (36) and (37), the equation (35) becomes

$$D^2 w\left[\left(1+\frac{k_x^2 V^2}{n^2}\right)^2+\frac{4\Omega^2}{n^2}\right]+\beta D w\left[\left(1+\frac{k_x^2 V^2}{n^2}\right)^2+\frac{4\Omega^2}{n^2}\right] \\ +w\left(1+\frac{k_x^2 V^2}{n^2}\right)\left[g\beta\frac{k^2}{n^2}-k^2\left(1+\frac{k_x^2 V^2}{n^2}\right)\right]=0, \tag{38}$$

where we have written $V^2 = \mu H_1^2/4\pi\rho_1$.

We shall now discuss some special cases before coming back to the general case.

(a) *Non-rotating configuration*

If we suppose that the configuration is non-rotating, equation (38) simplifies to

$$D^2 w+\beta D w+w k^2 \frac{\left[\frac{g\beta}{n^2}-\left(1+\frac{k_x^2 V^2}{n^2}\right)\right]}{\left[1+\frac{k_x^2 V^2}{n^2}\right]}=0. \tag{39}$$

This particular case of zero rotation has been treated before (Talwar 1959, hereafter called paper I) and the equation determining n is

$$n^2=\frac{4k^2 l^2 g\beta}{4(k^2 l^2+\alpha^2\pi^2)+\beta^2 l^2}-k_x^2 V^2, \tag{40}$$

where α is an integer. The equation (40) clearly manifests the stabilizing influence of horizontal magnetic field. Harmonic disturbances (when $\beta < 0$) give rise to horizontal waves propagated with phase velocity given by

$$U_{p,k}=\left[V_k^2-\frac{g\beta l^2}{k^2 l^2+\alpha^2\pi^2}\right]^{\frac{1}{2}}, \tag{41}$$

which is greater than the Alfvén velocity V_k in the direction **k** when $\beta < 0$. Here $|\beta l| \ll 1$, which means that the total density change in the fluid between $z = 0$ and $z = l$ is much less than the average density. The group velocity $U_{g,k}$ is written as,

$$U_{g,k} = \frac{[V_k^2(k^2l^2+\alpha^2\pi^2)^2 - g\beta l^2\alpha^2\pi^2]}{[V_k^2(k^2l^2+\alpha^2\pi^2) - g\beta l^2]^{\frac{1}{2}}\,(k^2l^2+\alpha^2\pi^2)^{\frac{3}{2}}}, \tag{42}$$

which is less than the phase velocity for $\beta < 0$. The equations (40) and (41) suggest that whereas under gravity alone there is complete instability for $\beta > 0$, there is no instability, in our present context, for wavelengths smaller than a certain critical value λ_* given by the following equation

$$k_* = \frac{2\pi}{\lambda_*} = \left[\frac{g\beta}{V_k^2} - \frac{\alpha^2\pi^2}{l^2}\right]^{\frac{1}{2}}. \tag{43}$$

Again (43) shows that the critical field necessary for complete stability (corresponding to the lowest value of the parameter α, namely unity) is given by

$$V_k^2 = g\beta l^2/\pi^2. \tag{44}$$

Further, since $n = 0$ at $k = 0$ and also at $k = k_*$, there should be a mode of maximum instability for an intermediate value of k given by

$$k_m = \left[\frac{(g\beta)^{\frac{1}{2}}\,\alpha\pi}{V_k l} - \frac{\alpha^2\pi^2}{l^2}\right]^{\frac{1}{2}} \tag{45}$$

and the growth rate, n_m, of this mode is

$$n_m = \frac{k_m^2\, l V_k}{\alpha\pi}. \tag{46}$$

For $k = k_*$ the wave velocity vanishes and the group velocity becomes infinitely large. It is interesting to note that whereas the wave velocity is more than the group velocity for $\beta < 0$ (ordinarily stable configuration), the group velocity becomes more than the wave velocity in the presence of a horizontal magnetic field for the case $\beta > 0$ (when the configuration is thoroughly unstable without magnetic field) in the region of wavelength 0 to λ_*. Thus we conclude, in our present context, that, for waves which travel with speed less than the Alfvén wave velocity, the group velocity is greater than the phase velocity, while for those with speeds greater than the Alfvén velocity the group velocity is less than the wave velocity.

(*b*) *Rotating configuration without magnetic field*

If the medium is field-free, and is partaking in a uniform rotation about the z-axis then the expression for n can be derived to be (when $|\beta l| \ll 1$),

$$n^2 = \frac{g\beta k^2l^2 - 4\Omega^2\alpha^2\pi^2}{k^2l^2+\alpha^2\pi^2}. \tag{47}$$

This equation shows that the coriolis force has a stabilizing influence on the configuration. For $\beta < 0$, the expressions for the phase and group velocities can be easily worked out. The equation (47) also suggests that even when $\beta > 0$ (when under pure gravity the configuration is thoroughly unstable) we may have stability provided

$$\Omega^2 > \frac{g\beta k^2 l^2}{4\alpha^2\pi^2}. \tag{48}$$

Thus for a given positive value of β, and for a given angular speed, (48) defines a critical value k^* of wave-number, namely $(4\Omega^2\alpha^2\pi^2/g\beta l^2)^{\frac{1}{2}}$. For disturbances with wave-numbers less than this critical, the configuration will be stable and will be unstable only for values of the wave-number greater than this. Thus, as in the corresponding case of magnetic field, we find that rotation has a stabilizing effect. There are, however, a few points of difference in the nature of stabilizing role of magnetic field and rotation. Coriolis force has greater stabilizing effect for large wavelength, whereas magnetic field stabilizes smaller wavelengths to a greater extent. Again there exists a critical magnetic field given by equation (44) which can remove instability altogether (for all k), but it seems that no finite amount of rotation can achieve that result. Also there does not seem to exist a maximum of the growth-rate of any mode at finite wavelength in the presence of rotation, although there does (*vide* equation 45) in the case of magnetic field. The maximum growth-rate occurs, in this case of zero magnetic field, at $k_m = \infty$ and is given by $n_m = (g\beta)^{\frac{1}{2}}$. The coriolis forces do not act on motion parallel to the axis of rotation, and the result that $k_m = \infty$ independently of rotation is associated with the fact that there is no horizontal motion when $k_m = \infty$.

(c) *Simultaneous presence of rotation and magnetic field*

If m_1, m_2 are the roots of the equation (38), then

$$m_1 + m_2 = -\beta \tag{49}$$

and $$m_1 m_2 = k^2\left[\frac{g\beta}{n^2} - \left(1+\frac{k_x^2 V^2}{n^2}\right)\right]\left[1+\frac{k_x^2 V^2}{n^2}\right]\left[\left(1+\frac{k_x^2 V^2}{n^2}\right)^2 + \frac{4\Omega^2}{n^2}\right]^{-1},$$

and the general solution of the equation (38) is

$$w(z) = A\,e^{m_1 z} + B\,e^{m_2 z}, \tag{50}$$

where A, B are constants to be determined from the boundary conditions. Since the fluid is bounded by rigid planes at $z = 0$, and $z = l$,

$$w = 0 \quad \text{at} \quad z = 0, \quad \text{which gives} \quad B = -A \tag{51}$$

and $$w = 0 \quad \text{at} \quad z = l, \quad \text{which gives} \quad e^{(m_1 - m_2)l} = 1, \tag{52}$$

whence $$(m_1 - m_2)\,l = 2\alpha i\pi, \tag{53}$$

α being an integer. Thus equation (50) may be rewritten as,

$$w(z) = A'\,e^{-\frac{1}{2}\beta z}\sin\frac{\alpha\pi z}{l}. \tag{54}$$

The equation for n for different values of the parameter α can then be worked out to be

$$n^4 + n^2\left[2k^2 V_k^2 + \frac{4\Omega^2 E - g\beta}{1+E}\right] + \left[k^4 V_k^4 - \frac{g\beta k^2 V_k^2}{1+E}\right] = 0, \tag{55}$$

where we have written E for $(4\alpha^2\pi^2+\beta^2l^2)/4k^2l^2$ for convenience, and V_k denotes the component of Alfvén velocity in the direction of wave-number **k**. The two roots for n^2 of this equation are both real and negative for $\beta < 0$ and also when β is positive but less than $(1+E).(k^2V_k^2/g)$. Thus we can conclude that a configuration of stratified fluid under the joint influence of a horizontal magnetic field and the coriolis force will be stable not only when the density is decreasing function of the vertical distance but even when $\beta > 0$ and less than $[(1+E)k^2V_k^2]/g$. The configuration will therefore be unstable only when $\beta > [(1+E)k^2V_k^2]/g$,

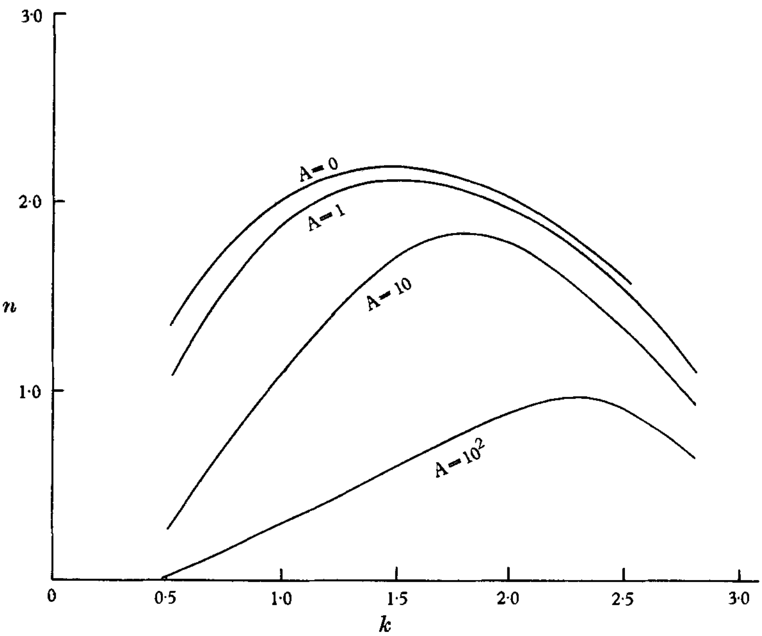

FIGURE 1. Illustrates the inhibiting influence of rotation in the unstable case of a continuously stratified fluid with horizontal magnetic field. The growth-rate n (measured with the unit $\pi V_k/l\,\sec^{-1}$) is plotted as a function of wave-number **k** (measured with the unit $\pi/l\,\text{cm}^{-1}$) for $B\ (= g\beta l^2/\pi^2V_k^2) = 10$ and for several values of $A\ (= 4\Omega^2l^2/\pi^2V_k^2)$.

since then one of the two roots of the above equation becomes positive. When the configuration is stable, harmonic disturbance leads to propagation of horizontal waves with velocity given by the following expression:

$$U_{p_{1,2}}^2 = \left[V_k^2 + \frac{4\Omega^2\alpha^2\pi^2/k^2 - g\beta l^2}{2(\alpha^2\pi^2+k^2l^2)}\right] \mp \left[\left\{\frac{4\Omega^2\alpha^2\pi^2/k^2 - g\beta l^2}{2(\alpha^2\pi^2+k^2l^2)}\right\}^2 + 4V_k^2\frac{\Omega^2}{k^2}\frac{\alpha^2\pi^2}{(\alpha^2\pi^2+k^2l^2)}\right]^{\frac{1}{2}}. \tag{56}$$

We may also write this expression as

$$U_{p_{1,2}}^2 - V_k^2 = \frac{4\Omega^2\alpha^2\pi^2/k^2 - g\beta l^2}{2(\alpha^2\pi^2+k^2l^2)}\left[1 \mp \left\{1 + \frac{16\Omega^2\alpha^2\pi^2V_k^2(k^2l^2+\alpha^2\pi^2)}{k^2[4\Omega^2(\alpha^2\pi^2/k^2) - g\beta l^2]^2}\right\}^{\frac{1}{2}}\right]. \tag{57}$$

We find therefore that simultaneous presence of rotation and field produces a stabilizing effect; as we have already seen, that rotation and magnetic field individually also have a stabilizing influence.

In the unstable case ($\beta > [(1+E)\,k^2 V_k^2]/g$) we are interested in the real positive root of equation (55). Let us measure k and n in units of $(\pi/l)\,\text{cm}^{-1}$ and $(\pi V_k/l)\,\text{sec}^{-1}$, respectively, so that the equation (55) can be rewritten in the form

$$n^4 + n^2\left[2k^2 + \frac{A\alpha^2 - Bk^2}{k^2+\alpha^2}\right] + k^4\left[1 - \frac{B}{k^2+\alpha^2}\right] = 0, \tag{58}$$

where
$$A = 4\Omega^2\alpha^2/\pi^2 V_k^2 \quad \text{and} \quad B = \frac{g\beta l^2}{\pi^2\, V_k^2} \tag{59}$$

are pure numbers.

The calculation of the positive root of the above equation has been carried out for $B = 10$ and for several values of A for the case $\alpha = 1$, and the results are presented graphically in figure 1. Curves of n against k for $B = 10$ and $A = 0, 1, 10, 10^2$ are given. These curves clearly show (i) that for a given k, n decreases with increase in A, (ii) that n_m, the maximum growth-rate, also decreases with increase of A, and (iii) that k_m, the wave-number for the mode of maximum instability increases with increasing A. The parameter A being a measure of the relative dynamical importance of the coriolis forces with the magnetic forces, we can conclude that, for a given B, the effect of increase in rotation is to increase the time taken for the system to depart from equilibrium and to decrease the wavelength of the mode of maximum instability.

4. Hydromagnetic stability of two superposed rotating fluids

In this section we shall investigate the effect of a uniform horizontal magnetic field on the equilibrium of two uniform, inviscid, infinitely conducting, superposed *rotating* fluids. The fluids will be taken to have uniform densities ρ_1 (for lower fluid) and ρ_2 (for upper fluid) and will be assumed to rotate uniformly about the z-axis.

In this particular case equation (35) reduces to

$$\left[n + \frac{\mu H^2}{4\pi\rho}\frac{k_x^2}{n}\right]^2 (D^2 - k^2)\,w + 4\Omega^2 D^2 w = 0, \tag{60}$$

since we have assumed $D\rho = 0$ and $DH = 0$.

The equation is true for either fluid. Hence we can write

$$w_1 = A_1 e^{m_1 z} + B_1 e^{-m_1 z} \quad (z < 0), \tag{61}$$

$$w_2 = A_2 e^{-m_2 z} + B_2 e^{m_2 z} \quad (z > 0). \tag{62}$$

The boundary conditions to be satisfied are as follows. (i) The velocity should vanish at $z = -\infty$ (for lower fluid) and at $z = +\infty$ (for upper fluid). (ii) The normal component of velocity should be continuous at the interface, i.e. to our approximation $w(z)$ is continuous at $z = 0$. (iii) The pressure should be continuous across the interface, which means that

$$\delta p_1 - \delta p_2 + g(\rho_2 - \rho_1)\,\xi + \frac{\mu H}{4\pi}\,[(h_x)_1 - (h_x)_2] = 0, \tag{63}$$

where the subscripts 1, 2 stand for the lower ($z < 0$) and upper fluids ($z > 0$), respectively. Here ξ denotes the displacement of the interface given by w/n.

On applying the boundary conditions (i) and (ii), we find that equations (61) and (62) become

$$w_1 = A\,e^{m_1 z} \quad (z < 0), \qquad w_2 = A\,e^{-m_2 z} \quad (z > 0), \tag{64}$$

since $A_1 = A_2 = A$.

On making use of the equations (21)–(23), (25) and (34) and simplifying, we find that the equation (63) gives the following equation for n:

$$gk(\rho_2-\rho_1) = \rho_1[(n^2+k^2V_{k_1}^2)^2+4\Omega^2n^2]^{\frac{1}{2}}+\rho_2[(n^2+k^2V_{k_2}^2)^2+4\Omega^2n^2]^{\frac{1}{2}}. \tag{65}$$

This equation reduces to Rayleigh's result, namely $n^2 = gk(\rho_2-\rho_1)/(\rho_1+\rho_2)$, when both the coriolis force and the magnetic field are absent. The above equation is an 8th-degree equation in n, and is too difficult to be discussed without making some simplifications. Let us therefore investigate some simple special cases before we come back to the general case when both rotation and the magnetic field are present simultaneously.

(i) Non-rotating superposed fluids with magnetic field ($\Omega = 0$, $H \neq 0$):

The equation for n is

$$n^2 = gk\frac{\rho_2-\rho_1}{\rho_1+\rho_2}-\frac{\mu H_k^2}{2\pi}\frac{k^2}{\rho_1+\rho_2}; \tag{66}$$

this case has been thoroughly discussed in paper I.

(ii) Rotating superposed fluids with zero magnetic field ($\Omega \neq 0$, $H = 0$). This is a purely hydrodynamic case originally solved by Bjerknes *et al.* (1933), and the expression for n is easily seen to be (cf. Hide 1956)

$$n^2\left(1+\frac{4\Omega^2}{n^2}\right)^{\frac{1}{2}} = gk\alpha, \quad \text{where} \quad \alpha = \frac{\rho_2-\rho_1}{\rho_1+\rho_2}$$

or alternatively,

$$n^2 = gk\alpha[(1+x^2)^{\frac{1}{2}}-x], \quad \text{where} \quad x = 2\Omega^2/gk\alpha. \tag{67}$$

When $\rho_2 > \rho_1$, α and x are both positive and the configuration is unstable as in the non-rotating case, but the effect of increase in rotation is to decrease the positive value of n for a definite positive α.

Alternatively, when $\rho_2 < \rho_1$, the situation is thoroughly stable, as is also the case with zero rotation, and the effect of rotation is to increase the stability of the configuration since it makes n^2 more negative.

(iii) Homogeneous fluid under the joint influence of rotation and field ($\rho_1 = \rho_2 = \rho$).

For a single fluid carrying a uniform horizontal magnetic field and rotating uniformly about the vertical axis, the equation for n is

$$n^4+n^2(4\Omega^2+2k^2V_k^2)+V_k^4k^4 = 0, \tag{68}$$

which gives

$$n_{1,2}^2 = -(2\Omega^2+k^2V_k^2)\pm 2\Omega(\Omega^2+k^2V_k^2)^{\frac{1}{2}}; \tag{69}$$

the two roots are always negative thus showing stability. Harmonic disturbances lead to the propagation of two stable modes of waves with phase velocities given by

$$U_{p_{1,2}} = V_k[x\pm(1+x^2)^{\frac{1}{2}}], \tag{70}$$

where x is written for Ω/kV_k. Thus we find that the effect of coriolis force is to split an Alfvén wave into two waves travelling with different wave-speeds, one greater and one less than the Alfvén velocity. A similar result for a different orientation of field and rotation axes was obtained by Lehnert (1954). The group velocities for the two modes can be easily evaluated and are found to be

$$U_{g_{1,2}} = \frac{U_{p_{1,2}}}{(1+x^2)^{\frac{1}{2}}[(1+x^2)^{\frac{1}{2}} \pm x]} = \frac{V_k}{(1+x^2)^{\frac{1}{2}}}, \tag{71}$$

wherefrom it follows that for both modes the group velocity is less than the Alfvén velocity. Also it follows from the above expressions (70), (71) that for the mode for which the phase velocity is more than the Alfvén velocity, the corresponding group velocity is less than the phase velocity, but for the other mode for which the phase velocity is less than the Alfvén velocity, the corresponding group velocity is more than the phase velocity.

(iv) Instability in superposed fluids under joint influence of rotation and field. Putting $\rho_2/\rho_1 = \gamma$, we can rewrite equation (65) as

$$gk(\gamma-1) = [(n^2+k^2V_{k_1}^2)^2+4\Omega^2n^2]^{\frac{1}{2}} + \gamma[\{n^2+k^2(V_{k_1}^2/\gamma)\}^2+4\Omega^2n^2]^{\frac{1}{2}}. \tag{72}$$

As the equation cannot be easily discussed without simplifications, let us be satisfied with the investigation of the characteristics of the mode of maximum instability (when $\rho_2 > \rho_1$) under two conditions, namely, the 'high' and 'low' rotation cases. Because there is no fixed length scale in the system, we require that the coriolis force exceeds or is less than the other forces operating, that is, that

$$\Omega \gtrless n_m, \quad \text{and} \quad \Omega \gtrless k_m V_{k_1}$$

according as we are dealing with 'high' or 'low' rotation.

(a) *Instability in the 'low rotation' case*

In this case the magnetic field influences the motion much more than the coriolis force. The solution in the limiting case of zero rotation has already been given (paper I) as

$$n_m^2 = \frac{\gamma-1}{2(\gamma+1)}gk_m, \quad \text{and} \quad k_m = \frac{(\gamma-1)g}{4V_{k_1}^2}. \tag{73}$$

We expect that introduction of low rotation should not lead to results very different from the results quoted above for $\Omega = 0$. Under the assumptions $n_m > \Omega$ and $k_m V_{k_1} > \Omega$, we get the following characteristic equation for n

$$n^2(\gamma+1) = [g(\gamma-1)k - 2k^2V_{k_1}^2]\left[1-\frac{2\Omega^2}{k^2V_{k_1}^2}\frac{1+\gamma^2}{1+\gamma}\right]. \tag{74}$$

On differentiating this expression with respect to k, and putting $dn/dk = 0$, we find that for the mode of maximum instability

$$k_m \approx \frac{g(\gamma-1)}{4V_{k_1}^2}\left(1+\frac{2\Omega^2}{k_m^2V_{k_1}^2}\frac{1+\gamma^2}{1+\gamma}\right), \tag{75a}$$

and

$$n_m^2 \simeq \frac{g(\gamma-1)}{2(\gamma+1)}k_m\left[1-\frac{4\Omega^2}{k_m^2V_{k_1}^2}\frac{1+\gamma^2}{1+\gamma}\right]$$

$$= (n_m^2)_0\left[1-\frac{2\Omega^2}{k_m^2V_k^2}\frac{1+\gamma^2}{1+\gamma}\right], \tag{75b}$$

where $(n_m)_0$ represents the growth rate of the mode of maximum instability for zero rotation as given by (73) above. The value of k_m occurring on the right sides of the above expressions may, to the first order of approximation, be taken to refer to the zero rotation case. Thus we see that the presence of slow rotation in superposed fluids permeated by a horizontal field decreases the growth rate of the mode of maximum instability and, at the same time, increases the wave-number of this mode.

(b) *Instability in the high rotation case*

When the magnetic field is zero, it easily follows that, under the assumption $n^2 < \Omega^2$,

$$n = \frac{gk(\gamma-1)}{2\Omega(\gamma+1)}, \tag{76}$$

to the first approximation. Thus the mode of maximum instability, under pure rotation of a configuration of inviscid superposed fluids, is characterized by $n_m = \infty$ and $k_m = \infty$. We are interested in investigating the influence of the introduction of a small magnetic field on this configuration. It is easily seen that with the assumptions $\Omega > n_m$ and $\Omega \gg k_m V_{k_1}$, the characteristic equation for n becomes

$$n\left[1+\frac{k^2V_{k_1}^2}{2\Omega^2(\gamma+1)}+\frac{n^2}{8\Omega^2}\right] = \frac{gk(\gamma-1)}{2\Omega(\gamma+1)}, \tag{77}$$

which to the first order of approximation yields the following results for the mode of maximum instability:

$$k_m^2 = [2\Omega^2(\gamma+1)]/V_{k_1}^2, \quad n_m^2 = \frac{g^2(\gamma-1)^2}{8V_{k_1}^2(\gamma+1)}. \tag{78}$$

These equations show that the effect of introduction of low magnetic field in a rotating configuration of two superposed fluids of zero viscosity is to decrease both the wave-number and the growth-rate of the mode of maximum instability.

REFERENCES

Bjerknes, V., Bjerknes, J., Solberg, H. & Bergeron, T. 1933 *Physikalische Hydrodynamik*. Berlin: Springer.

Chandrasekhar, S. 1955 *Proc. Camb. Phil. Soc.* **51**, 162.

Ferraro, V. C. A. & Plumpton, C. 1958 *Astrophys. J.* **127**, 459.

Hide, R. 1955 *Proc. Roy. Soc.* A, **233**, 376.

Hide, R. 1956 *Quart. J. Mech. Appl. Math.* **9**, 35.

Lehnert, B. 1954 *Astrophys. J.* **119**, 647.

Rayleigh, Lord 1883 *Proc. Lond. Math. Soc.* **14**, 170 (also *Scientific Papers* **2**, 200).

Talwar, S. P. 1959 *Z. Astrophys.* **47**, 161.